Oxford Cambridge and RSA

Thursday 22 June 2023 – Afternoon

A Level Further Mathematics B (MEI)

Y434/01 Numerical Methods

Time allowed: 1 hour 15 minutes

9972798312

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

1 You are given that $(x_1, y_1) = (0.9, 2.3)$ and $(x_2, y_2) = (1.1, 2.7)$.

The values of x_1 and x_2 have been **rounded** to **1** decimal place.

(a) Determine the range of possible values of $x_2 - x_1$. **[2]**

The values of y_1 and y_2 have been **chopped** to **1** decimal place.

(b) Determine the range of possible values of $y_2 - y_1$. **[2]**

You are given that $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

(c) Determine the range of possible values of m. **[2]**

(d) Explain why your answer to part **(c)** is much larger than your answer to part **(a)** and your answer to part **(b)**. **[1]**

2 A car tyre has a slow puncture. Initially the tyre is inflated to a pressure of 34.5 psi. The pressure is checked after 3 days and then again after 5 days. The time t in days and the pressure, P psi, are shown in the table below. You are given that the pressure in a car tyre is measured in pounds per square inch (psi).

t	0	3	5
P	34.5	29.4	27.0

The owner of the car believes the relationship between P and t may be modelled by a polynomial.

(a) Explain why it is not possible to use Newton's forward difference interpolation method for these data. **[1]**

(b) Use Lagrange's form of the interpolating polynomial to find an interpolating polynomial of degree 2 for these data. **[4]**

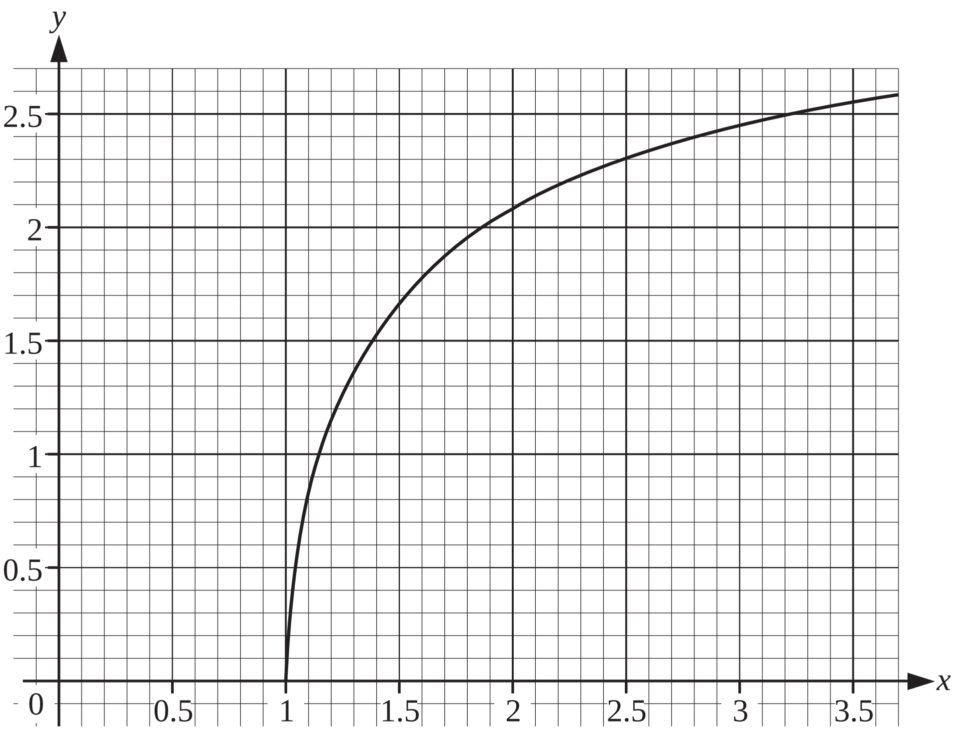
The car owner uses the polynomial found in part **(b)** to model the relationship between P and t.

Subsequently it is found that when $t = 6$, $P = 26.0$ and when $t = 10$, $P = 24.4$.

(c) Determine whether the owner's model is a good fit for these data. **[2]**

(d) Explain why the model would not be suitable in the long term. **[1]**

3 The diagram shows the graph of $y = \mathrm{f}(x)$ for values of x from 1 to 3.5.



The table shows some values of x and the associated values of y.

x	1.5	2	2.5
y	1.682137	2.094395	2.318559

(a) Use the forward difference method to calculate an approximation to $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$. **[2]**

(b) Use the central difference method to calculate an approximation to $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$. **[2]**

(c) On the copy of the diagram in the Printed Answer Booklet, show how the central difference method gives the approximation to $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$ which was found in part **(b)**. **[1]**

(d) Explain whether your answer to part **(a)** or your answer to part **(b)** is likely to give a better approximation to $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$. **[1]**

4 A spreadsheet is used to approximate $\int_a^b f(x)\,dx$ using the midpoint rule with 1 strip.

The output is shown in the table below.

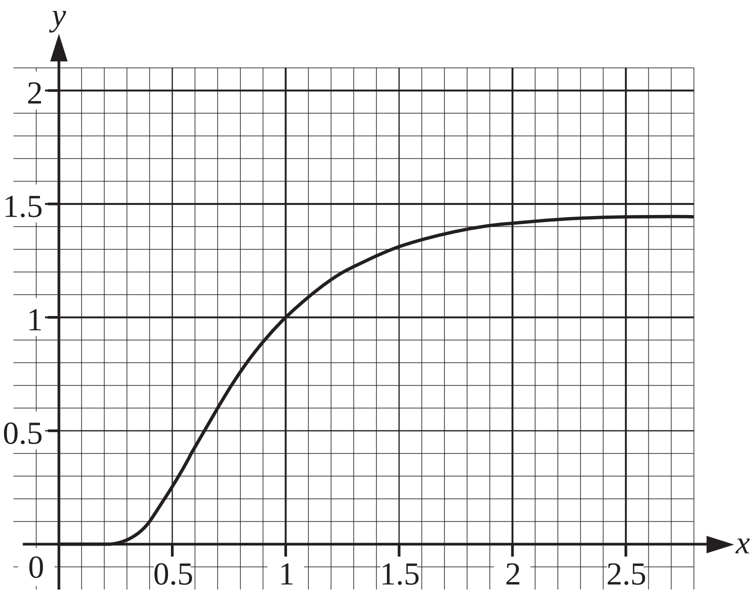
	B	C	D
3	x	$f(x)$	M_1
4	1.5	1.3103707	0.65518535

The formula in cell C4 is `=B4^(1/B4)`.

The formula in cell D4 is `=0.5*C4`.

(a) Write the integral in standard mathematical notation. **[3]**

A graph of $y = f(x)$ is included in the diagram below.

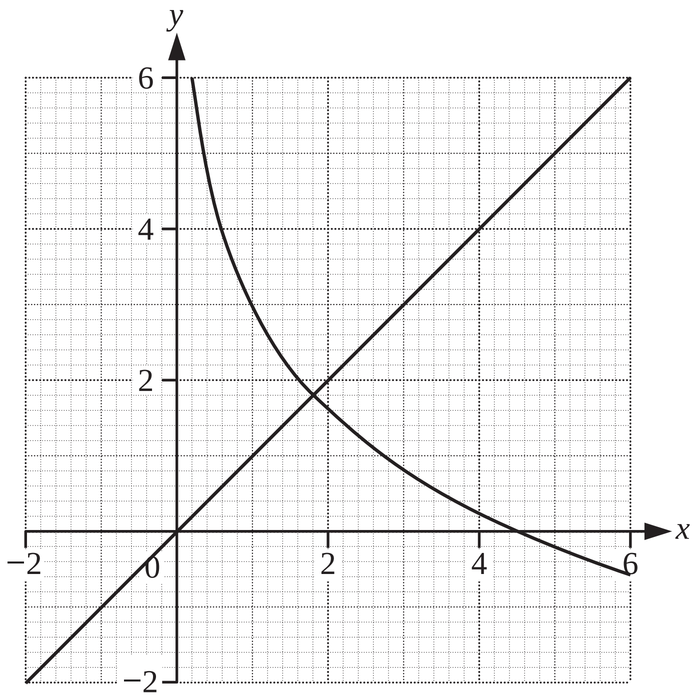


(b) Explain whether 0.65518535 is an over-estimate or an under-estimate of $\int_a^b f(x)\,dx$. **[1]**

5 The equation $3-2\ln x-x=0$ has a root near $x=1.8$.

A student proposes to use the iterative formula $x_{n+1}=\text{g}(x_n)=3-2\ln x_n$ to find this root.

The diagram shows the graphs of $y=x$ and $y=\text{g}(x)$ for values of x from -2 to 6.



(a) With reference to the graph, explain why it might not be possible to use the student's iterative formula to find the root near $x=1.8$. **[1]**

(b) Use the relaxed iteration $x_{n+1}=\lambda\text{g}(x_n)+(1-\lambda)x_n$, with $\lambda=0.475$ and $x_0=2$, to determine the root correct to **6** decimal places. **[3]**

A student uses the same relaxed iteration with the same starting value. Some analysis of the iterates is carried out using a spreadsheet, which is shown in the table below.

r	difference	ratio
0		
1	−0.1834898	
2	−0.0049137	0.02678
3	−6.44E-06	0.00131
4	−3.862E-09	0.0006
5	−2.313E-12	0.0006

(c) Explain what the analysis tells you about the order of convergence of this sequence of approximations. **[2]**

6 **(a)** **(i)** Calculate the relative error when π is **chopped** to **2** decimal places in approximating $\pi^2 + 2$. **[2]**

(ii) **Without** doing any calculation, explain whether the relative error would be the same when π is **chopped** to **2** decimal places when approximating $(\pi + 2)^2$. **[1]**

The table shows some spreadsheet output. The values of x in column A are exact.

	A	B	C
1	x	10^x	$\log_{10} 10^x$
2	1E-12	1	1.00001E-12
3	1E-11	1	9.99998E-12

The formula in cell B2 is `=10^A2`.

This has been copied down to cell B3.

The formula in cell C2 is `=LOG(B2)`.

This formula has been copied down to cell C3.

(b) **(i)** Write the value displayed in cell C2 in standard mathematical notation. **[1]**

(ii) Explain why the values in cells C2 and C3 are neither zero nor the same as the values in cells A2 and A3 respectively. **[2]**

7 The value of a function, $y = \mathrm{f}(x)$, and its gradient function, $\frac{\mathrm{d}y}{\mathrm{d}x}$, when $x = 2$, is given in **Table 7.1**.

Table 7.1

x	$\mathrm{f}(x)$	$\frac{\mathrm{d}y}{\mathrm{d}x}$
2	6	−2.8

(a) Determine the approximate value of the error when $\mathrm{f}(2)$ is used to estimate $\mathrm{f}(2.03)$. **[2]**

The Newton-Raphson method is used to find a sequence of approximations to a root, α, of the equation $\mathrm{f}(x) = 0$. The spreadsheet output showing the iterates, together with some further analysis, is shown in **Table 7.2**.

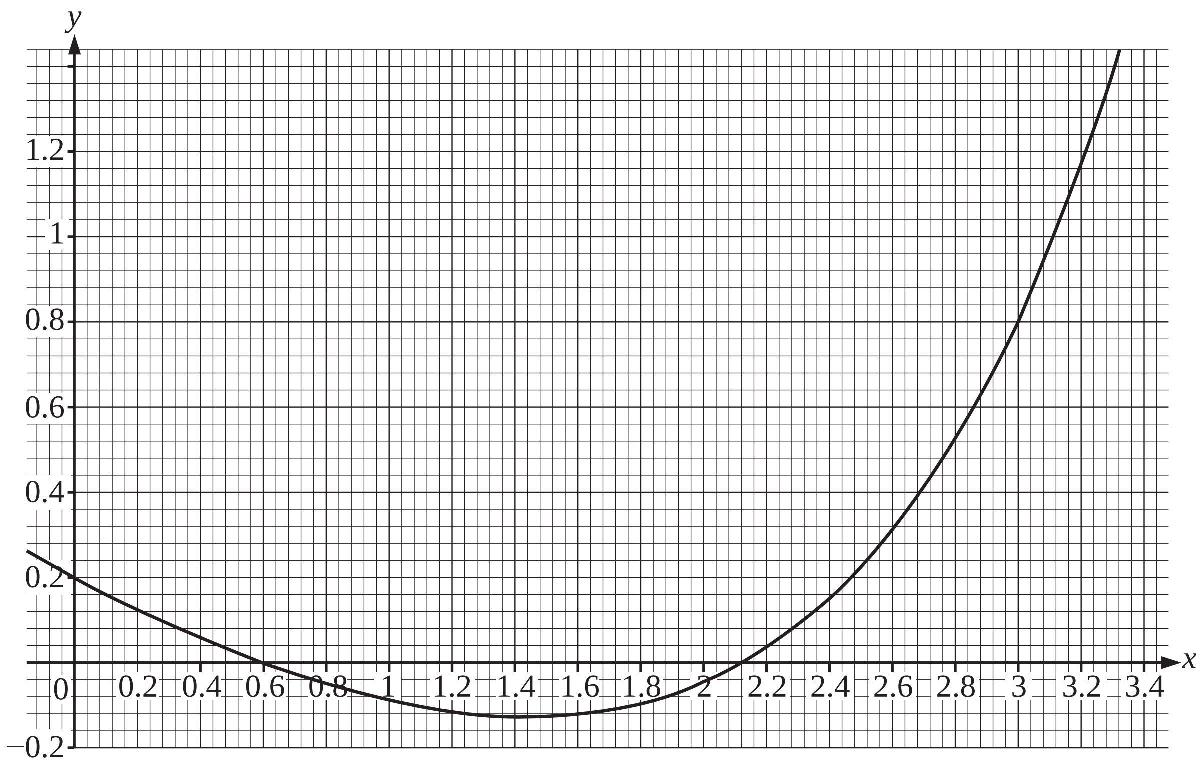
Table 7.2

	A	B	C	D
1	r	x_r	difference	ratio
2	0	12		
3	1	−13.1165572	−25.1165572	
4	2	1.76283279	14.87939004	−0.5924136
5	3	2.18052157	0.41768878	0.02807163
6	4	2.182419024	0.001897454	0.00454275
7	5	2.182419066	4.13985E-08	2.1818E-05

(b) **(i)** Explain what the values in column D tell you about the order of convergence of this sequence of approximations. **[2]**

(ii) **Without** doing any further calculation, state the value of α as accurately as you can, justifying the precision quoted. **[2]**

8 The graph of $y = 0.2\cosh x - 0.4x$ for values of x from 0 to 3.32 is shown on the graph below.



The equation $0.2\cosh x - 0.4x = 0$ has two roots, α and β where $\alpha < \beta$, in the interval $0 < x < 3$. The secant method with $x_0 = 1$ and $x_1 = 2$ is to be used to find β.

(a) On the copy of the graph in the Printed Answer Booklet, show how the secant method works with these two values of x to obtain an improved approximation to β. **[1]**

The spreadsheet output in the table below shows the result of applying the secant method with $x_0 = 1$ and $x_1 = 2$.

	I	J	K	L	M
2	r	x_r	$f(x_r)$	x_{r+1}	$f(x_{r+1})$
3	0	1	−0.0914	2	−0.0476
4	1	2	−0.0476	3.08529	0.95784
5	2	3.08529	0.95784	2.05134	−0.0298
6	3	2.05134	−0.0298	2.08259	−0.0181
7	4	2.08259	−0.0181	2.13042	0.00155
8	5	2.13042	0.00155	2.12664	−7E-05

(b) Write down a suitable cell formula for cell J4. **[1]**

(c) Write down a suitable cell formula for cell L4. **[2]**

(d) Write down the most accurate approximation to β which is displayed in the table. **[1]**

(e) Determine whether your answer to part **(d)** is correct to **5** decimal places. You should **not** calculate any more iterates. **[2]**

(f) It is decided to use the secant method with starting values $x_0 = 1$ and $x_1 = a$, where $a > 1$, to find α. State a suitable value for a. **[1]**

9 The trapezium rule is used to calculate 3 approximations to $\int_0^1 \sqrt[3]{\sinh(x)}\,dx$ with 1, 2 and 4 strips respectively. The results are shown in **Table 9.1**.

Table 9.1

n	T_n
1	0.52764369
2	0.66617652
4	0.72534275

(a) **Use these results** to determine **two** approximations to $\int_0^1 \sqrt[3]{\sinh(x)}\,dx$ using Simpson's rule. **[2]**

(b) Use your answers to part **(a)** to state the value of $\int_0^1 \sqrt[3]{\sinh(x)}\,dx$ as accurately as you can, justifying the precision quoted. **[1]**

Table 9.2 shows some further approximations found using the trapezium rule, together with some analysis of these approximations.

Table 9.2

n	T_n	difference	ratio
1	0.5276437		
2	0.6661765	0.138533	
4	0.7253427	0.059166	0.42709
8	0.7498821	0.024539	0.41475
16	0.7598858	0.010004	0.40766
32	0.7639221	0.004036	0.40348
64	0.7655404	0.001618	0.40095

(c) Explain what can be deduced about the order of the method in this case. **[2]**

(d) Use extrapolation to obtain the value of $\int_0^1 \sqrt[3]{\sinh(x)}\,dx$ as accurately as you can, justifying the precision quoted. **[4]**

END OF QUESTION PAPER

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